# Existence results for mean field equation with turbulence

Cheikh Birahim NDIAYE

SISSA, via Beirut 2-4, 34014 Trieste, Italy.

**Abstract**

In this paper we consider the following form of the so-called Mean field equation arising from the statistical mechanics description of two dimensional turbulence

$$(1) \qquad -\Delta_g u = \rho_1 \left( \frac{e^u}{\int_\Sigma e^u dV_g} - 1 \right) - \rho_2 \left( \frac{e^{-u}}{\int_\Sigma e^{-u} dV_g} - 1 \right)$$

on a given closed orientable Riemannian surface $(\Sigma, g)$ with volume $1$, where $\rho_1, \rho_2$ are real parameters. Exploiting the variational structure of the problem and running a min-max scheme introduced by Djadli and Malchiodi, we prove that if $k$ is a positive integer, $\rho_1$ and $\rho_2$ two real numbers such that $\rho_1 \in (8k\pi, 8(k+1)\pi)$ and $\rho_2 < 4\pi$ then $(1)$ is solvable.

## 1   Introduction

Many problems in physics can be formulated in terms of nonlinear elliptic equations with exponential nolinearities.

A typical example is the so called *mean field equation* on a given closed Riemannian surface $(\Sigma, g)$ with volume $1$.

$$(2) \qquad -\Delta_g u = \rho \left( \frac{h e^u}{\int_\Sigma h e^u dV_g} - 1 \right) \quad \text{on } \Sigma;$$

( where $\Delta_g$ is the Laplace-Beltrami, $\rho$ a real parameter) which arises in the study of limit of point vortices of Euler flows, spherical Onsager vortex theory and condensates in some Chern-Simons-Higgs models, see for example the papers [3], [4], [8], [9], [10], [11], [14], [25], [40] and the references therein.

An other example is what we refer as *mean field equation with turbulence* on a closed orientable Riemannian surface $(\Sigma, g)$

$$(3) \qquad -\Delta_g u = \rho_1 \left( \frac{e^u}{\int_\Sigma e^u dV_g} - \frac{1}{|\Sigma|} \right) - \rho_2 \left( \frac{e^{-u}}{\int_\Sigma e^{-u} dV_g} - \frac{1}{|\Sigma|} \right); \quad \int_\Sigma u dV_g = 0.$$

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[1]E-mail addresses: ndiaye@sissa

(where $\Delta_g$ is the Laplace-Beltrami, $|\Sigma|$ the volume of $\Sigma$ and $\rho_1$ and $\rho_2$ are two positive real parameters) arising from the statistical mechanics description of two-dimensional turbulence see Joyce and Montgomery [24] and Pointin and Lundgren [35].

The *mean field equation* (2) has received much attention in the last two decades. To mention some related non-trivial results, we cite the one of Ding-Jost-Li-Wang which asserts that if the underlying surface $\Sigma$ has positive genus then the problem has a solution provided $\rho \in ]8\pi, 16\pi[$, see [14]. Latter, using degree theory argument, Chen and Lin improve Ding-Jost-Li-Wang result by showing that if still the genus is positive then the problem is solvable for every $\rho \neq k8\pi$ where $k$ is an arbitrary positive integer, see [13]. Recently, Zindine Djadli refines Chen and Lin result by removing the constraint on the genus, see [15]. In the critical case, namely when $\rho = 8\pi$, Ding-Jost-Li-Wang have given sufficient conditions for the solvability.

From this panorama on the *mean field equation*, we see that the answer to the question of existence of solutions is quite satisfactory. However for the *mean field equation with turbulence* (3), little is known. To the best of our knowledge, the only available result in the literature is the one of Ohtsuka-Suzuki[34] and Ricciardi[36]. In fact Ohtsuka-Suzuki obtained existence of solutions for $\rho_i \in [0, 8\pi[$ via minimization and Ricciardi prove recently existence of Mountain-pass solutions under the assumptions that $(\Sigma, g)$ is a closed Riemannian surface such that the first non-zero eigenvalue $\mu_1(\Sigma)$ of $-\Delta_g$ verifies $8\pi < \mu_1(\Sigma)|\Sigma| < 16\pi$ and for $\rho_1$ and $\rho_2$ such that $\rho_1 + \rho_2 < \mu_1(\Sigma)|\Sigma|$ and $\max_{i=1,2}(\rho_i) > 8\pi$.

In this paper we will consider the following version of the *mean field equation with turbulence*

$$(4) \qquad -\Delta_g u = \rho_1 \left( \frac{e^u}{\int_\Sigma e^u dV_g} - 1 \right) - \rho_2 \left( \frac{e^{-u}}{\int_\Sigma e^{-u} dV_g} - 1 \right).$$

where $\Sigma$ has volume $1$ and the parameters $\rho_i$ are arbitrary real numbers (we recall that the relevant case for physics is when both are non-negative).

Prolem (4) is variational. Indeed critical points of the following functional

$$(5) \qquad \mathrm{II}_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 dV_g - \rho_1 \log \int_\Sigma e^{u-\bar{u}} dV_g - \rho_2 \log \int_\Sigma e^{-u+\bar{u}} dV_g, \quad u \in H^1(\Sigma);$$

(where $\rho = (\rho_1, \rho_2)$) are weak solutions, hence due to standard elliptic regularity are also classical solutions.

Our main goal is to give a more general existence result of the type of Zindine Djadli for the *mean field equation.*

We have indeed the following theorem.

**Theorem 1.1** *Suppose $k$ is a positive integer. Assume that $\rho_1 \in (k8\pi, 8(k+1)\pi)$ and $\rho_2 < 4\pi$, then problem (4) is solvable*

We are going to describe the main ideas in the proof of Theorem 1.1. From Theorem 2.1 below, if one of the $\rho_i$'s is bigger the $8\pi$, then the functional $\mathrm{II}_\rho$ is not bounded from below, hence extremals have to be found amongs saddle points. To do so we will use a min-max scheme introduced by Djadli and Malchiodi in their study of the existence of constant $Q$-curvature metrics on four manifolds, see [17]. By classical arguments in critical point theory, such a scheme yields existence of *Palais-smale* sequences, namely sequences $(u_l)$, such that

$$\mathrm{II}_\rho(u_l) \to c \in \mathbb{R}, \quad \mathrm{II}_\rho'(u_l) \to 0.$$

Since the functional $\mathrm{II}_\rho(\cdot)$ is invariant under translation by constant, then we can always assume that the sequence $(u_l)$ satisfies the normalization

$$\int_\Sigma e^{u_l} dV_g = 1 \quad \forall l.$$

If one proves that such sequences are bounded or that a similar compactness criterion holds, then the existence of solutions to problem (4) follows automatically.

To do so, we apply Struwe monotonicity method, see [38]. This consits in performing a min-max argument for for different values of $\rho$ of the form $\rho = (t\rho_1, t\rho_2)$ $t \sim 1$, and then to prove that there exists bounded Palais-Smale sequences for $II_\rho$ for $\rho = (t_l\rho_1, t_l\rho_2)$ with $t_l \to 1$. This yields existence of solutions to the problems.

$$-\Delta_g u = t_l\rho_1 \left( \frac{e^u}{\int_\Sigma e^u dV_g} - 1 \right) - t_l\rho_2 \left( \frac{e^{-u}}{\int_\Sigma e^{-u} dV_g} - 1 \right).$$

Hence an application of Proposition 2.5, gives the existence of solutions to problem (4).

From the discussion above, we have that the core of the analysis consist in finding Palais-Smale sequences. This will be done by characterizing the topology of low sublevels of the functional $II_\rho$. From considerations coming from an improvement of the Moser-Trudinger type inequality (Theorem 2.1), it follows that if $II_\rho(u)$ attains large negative values then $e^u$ has to concentrate near at most $k$ points of $\Sigma$. This means that, if we normalize $u$ so that $\int_\Sigma e^u dV_g = 1$, then naively $e^u \simeq \sum_{i=1}^k t_i \delta_{x_i}$, $x_i \in \Sigma$, $t_i \geq 0$, $\sum_{i=1}^k t_i = 1$. Such a family of convex combination of Dirac deltas are called formal barycenters of $\Sigma$ of order $k$, see Section 2, and will be denoted by $\Sigma_k$. With a further analysis (see Subsection 3.3 ), it is possible to show that the sublevel $\{II_\rho < -L\}$ for large $L$ has the same homology as $\Sigma_k$. Using the non-contractibility of $\Sigma_k$, we perform a min-max scheme, and get the Palais-Smale sequences.

## 2   Notation and preliminaries

In this section we collect some useful preliminary facts. For $x, y \in \Sigma$ we denote by $d(x, y)$ the metric distance between $x$ and $y$ on $\Sigma$. In the same way, we denote by $d(S_1, S_2)$ the distance between two sets $S_1, S_2 \subseteq \Sigma$, namely

$$d(S_1, S_2) = \inf \{d(x, y) \ : \ x \in S_1, y \in S_2\}.$$

Recalling that we are assuming $Vol_g(\Sigma) := \int_\Sigma 1 dV_g = 1$, given a function $u \in L^1(\Sigma)$, we denote its average (or integral) as

$$\overline{u} = \int_\Sigma u dV_g.$$

Below, by $C$ we denote large constants which are allowed to vary among different formulas or even within lines. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to $C$, as $C_\delta$, etc.. Also constants with subscripts are allowed to vary.

We now recall some Moser-Trudinger type inequalities and compactness results. The Euler-Langrange functional under study is the following

(6)      $$II_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 dV_g - \rho_1 \log \int_\Sigma e^{u-\overline{u}} dV_g - \rho_2 \log \int_\Sigma e^{-u+\overline{u}} dV_g, \quad u \in H^1(\Sigma);$$

which for large values of $\rho_1$ and $\rho_2$ will be in general unbounded from below. In fact, there is a precise criterion for $II_\rho$ to be bounded from below, which has been proved by Ohtsuka and Suzuki.

**Theorem 2.1** *([34]) For $\rho = (\rho_1, \rho_2)$ the functional $II_\rho$ is bounded from below if and only if both $\rho_1$ and $\rho_2$ satisfy the inequality $\rho_i \leq 8\pi$.*

Now we recall the following classical Moser-Trudinger inequality.

**Lemma 2.2** *There exists a constant $C_1$ depending only on $(\Sigma, g)$, such that*

$$\int_\Sigma e^{4\pi(u-\overline{u})} dV_g \leq C_1 \quad \forall u \in H^1(\Sigma) \ such \ that \ \int_\Sigma |\nabla u|^2 dV_g \leq 1.$$

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*As a consequence we have*

(7)
$$\log \int_\Sigma e^{(u-\overline{u})} dV_g \le C + \frac{1}{16\pi} \int_\Sigma |\nabla u|^2 dV_g, \quad \forall u \in H^1(\Sigma).$$

Next we give a compactness result due to Ohtsuka and Suzuki.

**Theorem 2.3** *([34]) Let $\rho_{1,n}$ and $\rho_{2,n}$ be sequences of non-negative real numbers satisfying*

$$\rho_{i,n} \to \rho_i \quad as \ n \to +\infty;$$

*and $u_n$ be a sequence of solutions to (4) corresponding to $(\rho_{i,n}, \rho_{2,n})$, with $\overline{u_n} = 0$. let also $\mu_{i,n}$ be the following Radon measures*

$$\mu_{1,n} = \frac{\rho_{1,n} e^{u_n}}{\int_\Sigma e^{u_n} dV_g} dV_g;$$

$$\mu_{2,n} = \frac{\rho_{2,n} e^{-u_n}}{\int_\Sigma e^{-u_n} dV_g} dV_g.$$

*Moreover let $w_{i,n}$ be as follows*

$$w_{i,n} = \int_\Sigma G(x,y) d\mu_{i,n}(y);$$

*where $G$ is the Green function of $-\Delta_g$ such that $\int_\Sigma G(\cdot,y) dV_g(y) = 0$, and we assume also without loss of generality that*

$$\mu_{i,n} \rightharpoonup \mu_i \ weakly*$$

*Let $S_1$ and $S_2$ denotes the following sets*

$$S_1 = \{x \in \Sigma : \ \exists x_n \in \Sigma \ s.t \ u_n(x_n) \to +\infty\};$$

*and*

$$S_2 = \{x \in \Sigma : \ \exists x_n \in \Sigma \ s.t \ u_n(x_n) \to -\infty\}.$$

*Then the following alternatives hold:*

*(1) ( Compactness)*
*We have that $S_1 \cup S_2 = \emptyset$ and there exists $u \in H^1(\Sigma), \overline{u} = 0$ and (up to subsequence)*

$$u_n \to u \ in \ H^1(\Sigma)$$

*and $u$ is a solution of (4) for $\rho_1$ and $\rho_2$.*

*(2) ( one-sided concentration)*
*There exists $i \in \{1,2\}$ such that $S_i \ne \emptyset$ and $S_j = \emptyset$ for $j \in \{1,2\} \setminus \{i\}$. Moreover, it holds that*

$$\mu_i = \sum_{x_0 \in S_i} 8\pi \delta_{x_0}$$

*and*

$$\mu_{i,n} \to 0 \ in \ L^\infty(\omega);$$

*for every $\omega \subset\subset \Sigma \setminus S_i$. On the other hand, there exists $w_j \in H^1(\Sigma)$ with $\overline{w_j} = 0$ such that up to a subsequence*

$$w_{j,n} \to w_j \ in \ H^1(\Sigma)$$

*and $w_j$ is solution to*

$$-\Delta_g w = l \left( \frac{Ke^w}{\int_\Sigma Ke^w dV_g} - 1 \right), \quad \overline{v} = 0;$$

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with $K(x) = e^{-\sum_{x_0 \in S_i} 8\pi G(x,x_0)}$.

(3) *(concentration)*

*For each $i = 1, 2$, we have $S_i \neq \emptyset$ and there exists a positive constant $m_i(x_0) \geq 4\pi$ for each $x_0 \in S_i$. Furthermore, we have a non-negative function $r_i \in L^1(\Sigma) \cap L^\infty_{loc}(M \setminus S_i)$ such that*

$$\mu_i = r_i + \sum_{x_0 \in S_i} m_i(x_0)\delta_{x_0};$$

*and*

$$\mu_{i,n} \to r_i \quad in \quad L^p(\omega)$$

*for every $p \in [0, +\infty[$ and every $\omega \subset\subset \Sigma \setminus S_i$. Finally the following fact hold:*

$3 - i)$

*If there exists $x_0 \in S_i \setminus S_j$ for $i \neq j$, then we have $m_i(x_0) = 8\pi$ and $r_i = 0$.*

$3 - ii)$

*For every $x_0 \in S_1 \cap S_2$, we have*

$$(m_1(x_0) - m_2(x_0))^2 = 8\pi(m_1(x_0) + m_2(x_0)).$$

*moreover, if $S_i \subset S_j$ and there exists $x_0 \in S_i$ satisfying*

$$m_i(x_0) - m_j(x_0) > 4\pi,$$

*then we have $r_i = 0$.*

Now we recall a Theorem due to Yanyan Li, which will be used to derive a compactness result adapted to our purposes.

**Theorem 2.4** *([26]) Let $(u_n)_n$ be a sequence of solutions of the equations*

$$-\Delta u_n = l_n \left( \frac{V_n e^{u_n}}{\int_\Sigma V_n e^{u_n} dV_g} - W_n \right),$$

*where $(V_n)_n$ and $(W_n)_n$ satisfy*

$$\int_\Sigma W_n dV_g = 1; \qquad \|W_n\|_{C^1(\Sigma)} \leq C; \qquad |\log V_n| \leq C; \qquad \|\nabla V_k\|_{L^\infty(\Sigma)} \leq C,$$

*and where $l_n \to l_0 > 0$, $l_0 \neq 8q\pi$ for $q = 1, 2, \dots$. Then, under the additional constraint $\int_\Sigma u_n dV_g = 0$, $(u_n)_n$ stays uniformly bounded in $L^\infty(\Sigma)$.*

Next we give a compactness result which describe all the possibles cases of Theorem 1.1.

**Proposition 2.5** *Let $K_1$ be a compact set of $\cup_{i=1}^\infty (8\pi i, 8\pi(i+1))$ and $K_2$ be a compact set of $(-\infty, 4\pi)$. Let $\rho_{1,n}$ be a sequence in $K_1$ and $\rho_{2,n}$ be a sequence in $K_2$. Moreover let $u_n$ be a sequence of solutions to (4) correspnding to $\rho_{1,n}$ and $\rho_{2,n}$ with $\bar{u}_n = 0$. Then we have $u_n$ is bounded in $C^m(\Sigma)$ for every positive integer $m$.*

PROOF. We first claim that for every $p > 1$ there exists $\bar{\rho}$ (depending on $K_1, K_2$ and $p$ )such that

(8) $$\int_\Sigma e^{-pu_n} dV_g \leq C.$$

To prove the Claim we use the Green representation formula for $-u_n$, an argument of Brezis and Merle, see [7] and thhe fact that $\rho_{1,n} > 0$. Indeed we have that

$$-u_n(x) \leq C + \int_\Sigma G(x, y) \left( 2\rho_{2,n} \frac{e^{-u_n}}{\int_\Sigma e^{-u_n} dV_g} \right) dV_g(y),$$

where $G(x, y)$ is the Green's function of $-\Delta_g$ on $\Sigma$. Next using Jensen's inequality we find

$$e^{-pu_n(x)} \leq C \int_\Sigma \exp(-2p\rho_{2,n} G(x, y)) \frac{e^{-u_n}}{\int_\Sigma e^{-u_n} dV_g} dV_g(y).$$

Now using the asymptotics of the Green function $(G(x, y) \simeq \frac{1}{2\pi} \log\left(\frac{1}{d(x,y)}\right))$ and also the Fubini theorem we get

$$\int_\Sigma e^{-pu_n} dV_g \leq C \sup_{x\in\Sigma} \int_\Sigma \frac{1}{d(x, y)^{\frac{p\rho_{2,n}}{\pi}}} dV_g(y).$$

Thus it is sufficient to take $\overline{\rho} = \frac{\pi}{2p}$ in order to obtain the claim.

Now suppose $\rho_{2,n} \geq \bar\rho$. Using Theorem 2.3 we have that three alternatives can occur. On the other hand since $\rho_{i,n} \in K_1$ and $\rho_{2,n}$, then it is trivially seen that the one-sided concentration and the concentration alternatives can not occur. Hence we have compactness and using standard elliptic regularity theory, we have boundedness in $C^m(\Sigma)$ for every $m$.

Now supose $\rho_{2,n} \leq \bar\rho$. Then from the Claim, we have $e^{-u_n}$ uniformly bounded in $L^p$. Hence $v_n$ defined as follows

$$(9) \qquad -\Delta_g v_n = -\rho_{2,n}\left(\frac{e^{-u_n}}{\int_\Sigma e^{-u_n} dV_g} - 1\right), \quad \bar{v}_n = 0$$

satisfies $v_n$ is uniformly bounded in $W^{2,p}$ (thanks to standard elliptic regularity). Thus taking $p$ so large we have by Sobolev-Embedding theorem $v_n$ is bounded in $C^{1,\alpha}$. Now defining $w_n$ by $w_n = u_n - v_n$, we have that $w_n$ sove the folowing PDE

$$(10) \qquad -\Delta_g w_n = \rho_{1,n}\left(e^{v_n} e^{v_n} \frac{e^{u_n}}{\int_\Sigma e^{v_n} e^{u_n} dV_g} - 1\right), \quad \bar{w}_n = 0$$

So using Theorem 2.4, we get $w_n$ is uniformly bounded in $L^\infty(\Sigma)$. Thus we get $u_n$ is uniformly bounded in $L^\infty(\Sigma)$. Hence, by standard elliptic regularity theory we get $u_n$ is bounded in $C^m(\Sigma)$ for every positive integer $m$. Hence the proposition is proved. ■

# 3   Proof of Theorem 1.1

This section deals with the proof of Theorem 1.1. It is divided into four subsections. The first one is concerned with the definition of the *formal barycenters* of $\Sigma$, and some related results. The second one is about the derivation of an improvement of the Moser-Trudinger type inequality given by Theorem 2.1 and corollaries. The third one deals with the construction of a continuous map from large negative sublevels of $\mathrm{II}_\rho$ into $\Sigma_k$ (for the definition see Subsection 1) and an other one from $\Sigma_k$ into suitable negative sublevels of $\mathrm{II}_\rho$. The last one describes the topological argument.

## 3.1   Barycenters and Properties

As said in the introduction of the Section, we start by recalling the definition of the so called *formal barycenters* of $\Sigma$.

For $k \in \mathbb{N}$, we let $\Sigma_k$ denote the family of formal sums

$$(11) \qquad \Sigma_k = \sum_{i=1}^k t_i \delta_{x_i}; \qquad t_i \geq 0, \quad \sum_{i=1}^k t_i = 1, \quad x_i \in \Sigma,$$

where $\delta_x$ stands for the Dirac delta at the point $x \in \Sigma$. We endow this set with the weak topology of distributions. This is known in literature as the *formal set of barycenters* of $\Sigma$ (of order $k$), see [1], [2], [5]. Although this is not in general a smooth manifold (except for $k = 1$), it is a *stratified set*, namely union

of cells of different dimensions. The maximal dimension is $3k - 1$, when all the points $x_i$ are distinct and all the $t_i$'s belong to the open interval $(0, 1)$.

After introducing the set of formal barycenters, we give the following well-know result (see [17]) which is necessary for the topological argument below.

**Lemma 3.1** *(well-known) For any $k \geq 1$ one has $H_{3k-1}(\Sigma_k; \mathbb{Z}_2) \neq 0$. As a consequence $\Sigma_k$ is non-contractible.*

Next we introduce a distance on $\Sigma_k$.
If $\varphi \in C^1(\Sigma)$ and if $\sigma \in \Sigma_k$, we denote the action of $\sigma$ on $\varphi$ as

$$\langle \sigma, \varphi \rangle = \sum_{i=1}^{k} t_i \varphi(x_i), \qquad \sigma = \sum_{i=1}^{k} t_i \delta_{x_i}.$$

Moreover, if $f$ is a non-negative $L^1$ function on $\Sigma$ with $\int_\Sigma f dV_g = 1$, we can define a distance of $f$ from $\Sigma_k$ in the following way

$$(12) \qquad dist(f, \Sigma_k) = \inf_{\sigma \in \Sigma_k} \sup \left\{ \left| \int_\Sigma f \varphi dV_g - \langle \sigma, \varphi \rangle \right| \mid \|\varphi\|_{C^1(\Sigma)} = 1 \right\}.$$

We also let

$$\mathcal{D}_{\varepsilon,k} = \left\{ f \in L^1(\Sigma) \ : \ f \geq 0, \|f\|_{L^1(\Sigma)} = 1, dist(f, \Sigma_k) < \varepsilon \right\}.$$

From a straightforward adaptation of the arguments of Proposition 3.1 in [17], we obtain the following result.

**Proposition 3.2** *Let $k$ be a positive integer, and for $\varepsilon > 0$ let $\mathcal{D}_{\varepsilon,k}$ be as above. Then there exists $\varepsilon_k > 0$, depending on $k$ and $\Sigma$ such that, for $\varepsilon \leq \varepsilon_k$ there exists a continuous map $\Pi_k : \mathcal{D}_{\varepsilon,k} \to \Sigma_k$.*

## 3.2 Improved Moser-Trudinger inequality and applications

In this subsection we analyze the Moser-Trudinger type inequality given by Theorem 2.1. We prove that depending on the amount of concentration of $e^u$ it get an improvement. From this we charcterizes low sublevels of $\mathrm{II}_\rho$ in terms of the concentration of $e^u$.

**Proposition 3.3** *Let $\delta_0 > 0$, $\ell \in \mathbb{N}$, and let $S_1, \ldots, S_\ell$ be subsets of $\Sigma$ satisfying $dist(S_i, S_j) \geq \delta_0$ for $i \neq j$. Let $\gamma_0 \in \left(0, \frac{1}{\ell}\right)$. Then, for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(\tilde{\varepsilon}, \delta_0, \gamma_0, \ell, \Sigma)$ such that*

$$\ell \log \int_\Sigma e^{(u-\overline{u})} dV_g + \log \int_\Sigma e^{-(u-\overline{u})} dV_g \leq C + \frac{1}{16\pi - \tilde{\varepsilon}} \int_\Sigma |\nabla u|^2 dV_g$$

*provided the function u satisfies the relations*

$$(13) \qquad \frac{\int_{S_i} e^u dV_g}{\int_\Sigma e^u dV_g} \geq \gamma_0, \qquad i \in \{1, \ldots, \ell\}.$$

Before making the proof we recall the following Lemma whose proof is a trivial adaptation of Lemma 3.2 in [31].

**Lemma 3.4** *Under the assumptions of Proposition 3.3, there exist numbers $\tilde{\gamma}_0, \tilde{\delta}_0 > 0$, depending only on $\gamma_0, \delta_0, \Sigma$, and $\ell$ sets $\tilde{S}_1, \ldots, \tilde{S}_\ell$ such that $d(\tilde{S}_i, \tilde{S}_j) \geq \tilde{\delta}_0$ for $i \neq j$ and such that*

$$\frac{\int_{\tilde{S}_1} e^u dV_g}{\int_\Sigma e^u dV_g} \geq \tilde{\gamma}_0, \quad \frac{\int_{\tilde{S}_1} e^u dV_g}{\int_\Sigma e^u dV_g} \geq \tilde{\gamma}_0; \qquad \frac{\int_{\tilde{S}_i} e^{-u} dV_g}{\int_\Sigma e^{-u} dV_g} \geq \tilde{\gamma}_0, \quad i \in \{2, \ldots, \ell\}.$$

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PROOF OF PROPOSITION 3.3. We use the argument in [17] adapted to our purpose. . Firts of all let $\tilde{S}_1, \ldots, \tilde{S}_\ell$ be given by Lemma 3.4. Moreover without loss of generality we assume that $\overline{u} = 0$. We have there exist $\ell$ functions $g_1, \ldots, g_\ell$ satisfying the properties

(14)
$$\begin{cases} g_i(x) \in [0,1] & \text{for every } x \in \Sigma; \\ g_i(x) = 1, & \text{for every } x \in \tilde{S}_i, i = 1, \ldots, \ell; \\ supp(g_i) \cap supp(g_j) = \emptyset, & \text{for } i \neq j; \\ \|g_i\|_{C^2(\Sigma)} \leq C_{\tilde{\delta}_0}, \end{cases}$$

where $C_{\tilde{\delta}_0}$ is a positive constant depending only on $\tilde{\delta}_0$.

Next we decompose the function $u$ in Fourier mode (to be choosen later) as follows

(15)
$$u = \hat{u} + \tilde{u}; \qquad \hat{u} \in L^\infty(\Sigma).$$

Now using Lemma 3.4, for any $b \in 2, \ldots, \ell$ we can write that

$$\begin{aligned} \ell \log \int_\Sigma e^u dV_g + \log \int_\Sigma e^{-u} dV_g &= \log \left[ \left( \int_\Sigma e^u dV_g \int_\Sigma e^{-u} dV_g \right) \left( \int_\Sigma e^u dV_g \right)^{\ell-1} \right] \\ &\leq \left[ \left( \int_{\tilde{S}_1} e^u dV_g \int_{\tilde{S}_1} e^{-u} dV_g \right) \left( \int_{\tilde{S}_b} e^u dV_g \right)^{\ell-1} \right] - \ell \log \tilde{\gamma}_0 \\ &\leq \log \left[ \left( \int_\Sigma e^{g_1 u} dV_g \int_\Sigma e^{-g_1 u} dV_g \right) \left( \int_\Sigma e^{g_b u} dV_g \right)^{\ell-1} \right] \\ &\quad - \ell \log \tilde{\gamma}_0, \end{aligned}$$

Using the fact that $\hat{u}$ belong to $L^\infty(\Sigma)$, wearrive to

$$\begin{aligned} \ell \log \int_\Sigma e^u dV_g + \log \int_\Sigma e^{-u} dV_g &\leq \log \left[ \left( \int_\Sigma e^{g_1 \tilde{u}} dV_g \int_\Sigma e^{-g_1 \tilde{u}} dV_g \right) \left( \int_\Sigma e^{g_b \tilde{u}} dV_g \right)^{\ell-1} \right] \\ &\quad - \ell \log \tilde{\gamma}_0 + (\ell+1) \|\hat{u}\|_{L^\infty(\Sigma)}. \end{aligned}$$

Thus we get

$$\begin{aligned} \ell \log \int_\Sigma e^u dV_g + \log \int_\Sigma e^u dV_g &\leq \log \int_\Sigma e^{g_1 \tilde{u}} dV_g + \log \int_\Sigma e^{-g_1 \tilde{u}} dV_g + (\ell-1) \int_\Sigma e^{g_b \tilde{u}} dV_g \\ (16) &\quad - \ell \log \tilde{\gamma}_0 + (1+\ell) \|\hat{u}\|_{L^\infty(\Sigma)}. \end{aligned}$$

Now apply Theorem 2.1 with parameters $(8\pi, 8\pi)$ to the couple $(g_1 \tilde{u}, -g_1 \tilde{u})$, and the standard Moser-Trudinger inequality (7) to $g_b \tilde{u}$ we obtain

$$\log \int_\Sigma e^{g_1 \tilde{u}} dV_g + \log \int_\Sigma e^{-g_1 \tilde{u}} dV_g \leq \frac{1}{16\pi} \int_\Sigma |\nabla g_1 u|^2 dV_g + C;$$

$$(\ell-1) \int_\Sigma e^{g_b \tilde{u}} dV_g \leq \frac{(\ell-1)}{16\pi} \int_\Sigma |\nabla(g_b \tilde{u})|^2 dV_g + (\ell-1)\overline{g_b \tilde{u}} + (\ell-1)C.$$

Putting together (16)-(17) we get

$$\ell \log \int_\Sigma e^u dV_g + \log \int_\Sigma e^{-u} dV_g \leq \frac{1}{16\pi} \int_\Sigma |\nabla g_1 u|^2 dV_g + \frac{(\ell-1)}{16\pi} \int_\Sigma |\nabla(g_b \tilde{u})|^2 dV_g + (\ell-1)\overline{g_b \tilde{u}} + C.$$

Next, by interpolation, for any $\varepsilon > 0$ there exists $C_{\varepsilon, \tilde{\delta}_0}$ (depending only on $\varepsilon$ and $\tilde{\delta}_0$) such that

(17)
$$\frac{1}{16\pi} \int_\Sigma |\nabla(g_i \tilde{u})|^2 dV_g \leq \frac{1}{16\pi} \int_\Sigma g_i^2 |\nabla \tilde{u}|^2 dV_g + \frac{\epsilon}{16\pi} \int_\Sigma |\nabla \tilde{u}|^2 dV_g + C_{\varepsilon, \tilde{\delta}_0} \int_\Sigma \tilde{u}^2 dV_g.$$

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Hence inserting this inequality into (17) we get

$$\ell \log \int_\Sigma e^u dV_g + \log \int_\Sigma e^{-u} dV_g \leq \frac{1}{16\pi} \int_\Sigma g_1^2 |\nabla \tilde{u}|^2 dV_g + \frac{(\ell-1)}{16\pi} \int_\Sigma g_b^2 |\nabla \tilde{u}|^2 dV_g + \frac{\ell\epsilon}{16\pi} \int_\Sigma |\nabla \tilde{u}|^2 dV_g$$
$$+ \ell C_{\varepsilon,\tilde\delta_0} \int_\Sigma \tilde{u}^2 dV_g + (\ell-1)\overline{g_b \tilde{u}} + C,$$

Now for $b = 2, \ldots, \ell$, we choose $b \in \{2, \ldots, \ell\}$ such that

$$\frac{1}{16\pi} \int_\Sigma g_b^2 |\nabla \tilde{u}|^2 dV_g \leq \frac{1}{\ell-1} \frac{1}{16\pi} \int_{\cup_{s=2}^\ell supp(g_s)} |\nabla u|^2 dV_g.$$

On the other hand since the $g_i's$ have disjoint supports, see (14), then last formula yields

$$\ell \log \int_\Sigma e^u dV_g + \log \int_\Sigma e^{-u} dV_g \leq \frac{1+\ell\epsilon}{16\pi} \int_\Sigma |\nabla \tilde{u}|^2 dV_g + \ell C_{\varepsilon,\tilde\delta_0} \int_\Sigma \tilde{u}^2 dV_g + (\ell-1)\overline{g_b \tilde{u}} + C$$

Next, by elementary estimates we find

$$\ell \log \int_\Sigma e^u dV_g + \log \int_\Sigma e^{-u} dV_g \leq \frac{1+\ell\epsilon}{16\pi} \int_\Sigma |\nabla \tilde{u}|^2 dV_g + C_{\varepsilon,\tilde\delta_0,\ell} \int_\Sigma \tilde{u}^2 dV_g + C_{\varepsilon,\tilde\delta_0,\ell,\tilde\gamma_0} + \ell\|\hat{u}\|_{L^\infty(\Sigma)}.$$

Now comes the choice of $\hat{u}$, see (15). We choose $\tilde{C}_{\varepsilon,\tilde\delta_0,\ell}$ to be so large that the following property holds

$$C_{\varepsilon,\tilde\delta_0,\ell} \int_\Sigma v^2 dV_g < \frac{\epsilon}{16\pi} \int_\Sigma |\nabla v|^2 dV_g, \qquad \forall v \in V_{\varepsilon,\tilde\delta_0,\ell},$$

where $V_{\varepsilon,\tilde\delta_0,\ell}$ denotes the span of the eigenfunctions of the Laplacian on $\Sigma$ corresponding to eigenvalues bigger than $\tilde{C}_{\varepsilon,\tilde\delta_0,\ell}$.

Then we set

$$\tilde{u} = P_{V_{\varepsilon,\tilde\delta_0,\ell}} u; \qquad \hat{u} = P_{V_{\varepsilon,\tilde\delta_0,\ell}^\perp} u,$$

where $P_{V_{\varepsilon,\tilde\delta_0,\ell}}$ (resp. $P_{V_{\varepsilon,\tilde\delta_0,\ell}^\perp}$) stands for the orthogonal projection onto $V_{\varepsilon,\tilde\delta_0,\ell}$ (resp. $V_{\varepsilon,\tilde\delta_0,\ell}^\perp$). Since $\overline{u} = 0$, the $H^1$-norm and the $L^\infty$-norm on $V_{\varepsilon,\tilde\delta_0,\ell}^\perp$ are equivalent (with a proportionality factor which depends on $\varepsilon, \tilde\delta_0$ and $\ell$), hence by our choice of $u$ there holds

$$\|\hat{u}\|_{L^\infty(\Sigma)}^2 \leq \hat{C}_{\varepsilon,\tilde\delta_0,\ell} \frac{1}{16\pi} \int_\Sigma |\nabla \hat{u}|^2 dV_g dV_g; \qquad C_{\varepsilon,\tilde\delta_0,\ell} \int_\Sigma \tilde{u}^2 dV_g < \frac{\varepsilon}{16\pi} \int_\Sigma |\nabla \tilde{u}|^2 dV_g.$$

Hence the last formulas imply

$$\ell \log \int_\Sigma e^u dV_g + \log \int_\Sigma e^{-u} dV_g \leq \frac{1}{16\pi}(1+3\ell\varepsilon) \int_\Sigma |\nabla u|^2 dV_g + \hat{C}_{\varepsilon,\tilde\delta_0,\ell,\tilde\gamma_0}.$$

This concludes the proof. ∎

In the remaining of this subsection we will apply the above Proposition to understand the structure of the sublevels of $II_\rho$. Before this we state a Lemma which gives sufficient conditions for the improvement to hold. Its proof can be found in [17].

**Lemma 3.5** *Let $f \in L^1(\Sigma)$ be a non-negative function with $\|f\|_{L^1(\Sigma)} = 1$. We also fix an integer $\ell$ and suppose that the following property holds true. There exist $\varepsilon > 0$ and $r > 0$ such that*

$$\int_{\cup_{i=1}^\ell B_r(p_i)} f dV_g < 1 - \varepsilon \qquad \text{for all the } \ell\text{-tuples } p_1, \ldots, p_\ell \in \Sigma.$$

*Then there exist $\overline{\varepsilon} > 0$ and $\overline{r} > 0$, depending only on $\varepsilon, r, \ell$ and $\Sigma$ (and not on $f$), and $\ell+1$ points $\overline{p}_1, \ldots, \overline{p}_{\ell+1} \in \Sigma$ (which depend on $f$) satisfying*

$$\int_{B_{\overline{r}}(\overline{p}_1)} f dV_g > \overline{\varepsilon}, \ldots, \int_{B_{\overline{r}}(\overline{p}_{\ell+1})} f dV_g > \overline{\varepsilon}; \qquad B_{2\overline{r}}(\overline{p}_i) \cap B_{2\overline{r}}(\overline{p}_j) = \emptyset \text{ for } i \neq j.$$

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**Proposition 3.6** *Suppose $\rho_1 \in (8\pi k, 8\pi(k+1))$ and that $\rho_2 < 8\pi$. Then for any $\varepsilon > 0$ and any $r > 0$ there exists a large positive $L = L(\varepsilon, r)$ such that for every $u \in H^1(\Sigma)$ with $II_\rho(u) \le -L$ and with $\int_\Sigma e^u dV_g = 1$, there exists $k$ points $p_{1,u}, \ldots, p_{k,u} \in \Sigma$ such that*

$$(18) \qquad \int_{\Sigma \setminus \cup_{i=1}^k B_r(p_{i,u})} e^u dV_g < \varepsilon.$$

PROOF.  To prove the proposition. we willl argue by contradiction. So suppose it does not holds, then aplying Lemma 3.5 with $l = k$ and $f = e^u$, we have that there exists $\delta_0$, $\gamma_0$ and set $S_1, \cdots, S_{l+1}$ such that $d(S_i, S_j) \ge \delta_0$ and

$$(19) \qquad \frac{\int_{S_i} e^u dV_g}{\int_\Sigma e^u dV_g} \ge \gamma_0, \quad \text{for} \quad i = 1, \cdots, l+1.$$

Next from Jensen's inequality and the fact that $\int_\Sigma e^u dVg = 1$ we get

$$(20) \qquad \bar{u} \le 0 \quad \text{and} \quad \log \int_\Sigma e^{-u+\bar{u}} dV_g \ge 0.$$

Now since $\rho_1 < 8\pi(k+1)$ and $\rho_2 < 4\pi$ then there exits a small $\tilde{\epsilon} > 0$ such that

$$(21) \qquad (16\pi - \tilde{\epsilon})(k+1) > 2\rho_1 \quad \text{and} \quad 16\pi - \tilde{\epsilon} > 2\rho_2.$$

On the other hand from the definition of $II_\rho$ we have that

$$(22) \qquad \begin{aligned} II_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 dV_g &- (\frac{16\pi - \tilde{\epsilon}}{2})(k+1) \log \int_\Sigma e^{u-\bar{u}} dV_g - (\frac{16\pi - \tilde{\epsilon}}{2}) \log \int_\Sigma e^{-u+\bar{u}} dV_g \\ &+ (\frac{16\pi - \tilde{\epsilon}}{2})(k+1) - \rho_1) \log \int_\Sigma e^{u-\bar{u}} dV_g + (\frac{16\pi - \tilde{\epsilon}}{2} - \rho_2) \log \int_\Sigma e^{-u+\bar{u}} dV_g. \end{aligned}$$

Hence using (20), the normalizatiuon $\int_\Sigma e^u dV_g = 1$ and (21), we get

$$(23) \qquad II_\rho(u) \ge \frac{1}{2} \int_\Sigma |\nabla u|^2 dV_g - (\frac{16\pi - \tilde{\epsilon}}{2})(k+1) \log \int_\Sigma e^{u-\bar{u}} dV_g - (\frac{16\pi - \tilde{\epsilon}}{2}) \log \int_\Sigma e^{-u+\bar{u}} dV_g.$$

Next using (3.3) we obtain

$$(24) \qquad II_\rho(u) \ge -C$$

Hence the proposition is proved. ∎

The next result is a direct corollary of Proposition 3.6. It gives the distance of $e^u$ from $\Sigma_k$ for $u$ belonging to low sublels of $II_\rho$ and $\int_\Sigma e^u dV_g = 1$

**Corollary 3.7** *Let $\bar{\varepsilon}$ be a (small) arbitrary positive number and $k$ be given as in Theorem 1.1. Then there exists $L > 0$ such that, if $II(u) \le -L$ and $\int_\Sigma e^u dV_g = 1$, then we have that $d(e^u, \Sigma_k) \le \bar{\varepsilon}$.*

PROOF.   Let $\epsilon > 0$, $r > 0$ (to be fixed later) and let $L$ be the corresponding constant given by Proposition 3.6. We let $p_1, \cdots, p_k$ be the points given by Proposition 3.6 and we define $\sigma \in \Sigma_k$ as follows

$$(25) \quad \sigma = \sum_{i=1}^k t_i \delta_{p_i} \text{ where } t_i = \int_{A_{r,i}} e^{4u} dV_j, \quad A_{r,i} := B_{p_i}(r) \setminus \cup_{s=1}^{i-1} B_{p_s}(r), \ i = 1, \cdots, k-1, \ t_k = 1 - \sum_{i=1}^{k-1} t_i.$$

By construction we have $A_{r,i}$ are disjoint and $\cup_{i=1}^{k-1} A_{r,i} = \cup_{i=1}^{k-1} B_{p_i}(r)$. Now let $\varphi \in C^1(\Sigma)$ be such that $||\varphi||_{C^1(\Sigma)} = 1$, we have that by triangle inequality

$$(26) \qquad \left| \int_\Sigma e^u \varphi - <\sigma, \varphi> \right| \le \sum_{i=1}^{k-1} \left| \int_{A_{r,i}} e^u(\varphi - \varphi(p_i)) \right| + \left| \int_{\Sigma \setminus \cup_{i=1}^{k-1} A_{r,i}} e^u(\varphi -, \varphi(p_k)) \right|.$$

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Thus by using Mean value formula and (18) we get

$$(27) \qquad \left| \int_\Sigma e^{4u}\varphi - <\sigma,\varphi> \right| \le C_\Sigma r + C_\Sigma r\epsilon.$$

So by choosing $\epsilon$ and $r$ so small that $C_\Sigma r + C_\Sigma r\epsilon < \bar{\epsilon}$, and recalling that $d$ is the metric given by $C^1(\Sigma)^*$, we obtain

$$(28) \qquad d(e^u, \Sigma_k) < \bar{\epsilon};$$

hence we are done. ■

## 3.3 Construction of the projections $\Psi$ and $\Phi$

In this Subsection we construct two global continuous non-trivial projections in order to show that large negative sublevels of $\mathrm{II}_\rho$ have the same homology as $\Sigma_k$, see Proposition 3.9 below.

**Proposition 3.8** *Let $k$, $\rho_1$ and $\rho_2$ as in Theorem (1.1). Then there exists a large $L > 0$ and a continuous projection $\Psi$ from $\{\mathrm{II}_\rho \le -L\} \cap \{\int_\Sigma e^u dV_g = 1\}$ (with the natural topology of $H^1(\Sigma)$ ) onto $\Sigma_k$ which is homotopically non-trivial.*

PROOF. We fix $\varepsilon_k$ so small that Proposition 3.2 applies . Then we apply Corollary 3.7 with $\bar{\varepsilon} = \varepsilon_k$. We let $L$ be the corresponding large number, so that if $\mathrm{II}_\rho(u) \le -L$, then $dist(e^u, \Sigma_k) < \varepsilon_k$. Hence for these ranges of $u$ , since the map $u \mapsto e^u$ is continuous from $H^1(\Sigma)$ into $L^1(\Sigma)$, setting $\Psi(u) = \Pi_k(e^u)$ (where $\Pi_k$ is given by Proposition 3.2), we have $\Psi(\cdot)$ is continuous. The non-triviality of this map is a consequence of Proposition 3.9 **(ii)**. ■

Next, we show that one can map $\Sigma_k$ into very large negative sublevels of $\mathrm{II}_\rho$. To do this we start by introducing some notations..
Given $\sigma = \sum_{i=1}^k t_i \delta_{x_i} \in \Sigma_k$ and $l$ a positive real number, we set

$$(29) \qquad \varphi_{\sigma,l}(y) = \log \sum_{1=1}^k \left( \frac{l}{1+l^2 d_i(y)^2} \right)^2 - \log \pi, \quad y \in \Sigma;$$

where $d_i(y) = d(y, x_i)$.
We remark that, since the distance function is lipschitz, then $\varphi_{\sigma,l}$ is, hence due to Sobolev embedding is an element of $H^1(\Sigma)$.
We have the following Proposition about $\varphi_{\sigma,l}$.

**Proposition 3.9** *Supposs $k$, $\rho_1$ and $\rho_2$ as in Theorem 1.1. For $l > 0$ and $\sigma \in \Sigma_k$ we define*

$$\Phi_l : \Sigma_k \to H^1(\Sigma)$$

*as*

$$\Phi_l(\sigma) = \varphi_{\sigma,l}$$

*where $\varphi_{\sigma,l}$ is as in (29). Then for $L$ suficciently the exist $\bar{l} > 0$ such that*
  *(i) $\mathrm{II}_\rho(\Phi_l(\sigma)) \le -L$ uniformly in $\sigma \in \Sigma_k$ $l \ge \bar{l}$;*

*(ii) $\Psi \circ \Phi_l$ is homotopic to the identity on $\Sigma_k$ for $l$ large..*

PROOF. To prove $(i)$,we first claim that as $l \to +\infty$ the following estimate holds

$$(30) \qquad \int_\Sigma \varphi_{\sigma,l} = -2(1+o_l(1))\log l,$$

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$$(31) \qquad \log \int_{\Sigma} e^{\varphi_{\sigma,l}} dV_g = O(1) \ \text{ and } \ \log \int_{\Sigma} e^{-\varphi_{\sigma,l}} dV_g = 2(1 + o_l(1)) \log l,$$

and

$$(32) \qquad \int_{\Sigma} |\nabla \varphi_{\sigma,l}|^2 dV_g \le 32k\pi(1 + o_l(1)) \log l; .$$

**Proof of Claim**
*Proof of* (30)
Let $\delta \in (0, diam(\Sigma))$ be small. We have that

$$(33) \qquad 2\log \frac{l}{1 + l^2 diam(\Sigma)^2} - \log \pi \le \varphi_{\sigma,l} \le 2\log \frac{l}{1 + l^2 \delta^2} - \log \pi \quad \text{for } y \in \Sigma \setminus \cup_{1=1}^{k} B_{x_i}(2\delta);$$

and

$$(34) \qquad 2\log \frac{l}{1 + 4l^2 \delta^2} - \log \pi \le \varphi_{\sigma,l} \le 2\log l - \log \pi \quad \text{for } y \in \cup_{1=1}^{k} B_{x_i}(2\delta);$$

Now rewritting (**??**) we obtain

$$-2\log l - 2\log \left(1 + \frac{diam(\Sigma)^2}{l^2}\right) - \log \pi \le \varphi_{\sigma,l} \le -2\log l - 2\log \left(1 + \frac{\delta^2}{l^2}\right) - \log \pi \ \in \Sigma \setminus \cup_{1=1}^{k} B_{x_i}(2\delta)$$

Thus combining all, get

$$(35) \qquad \int_{\Sigma} \varphi_{\sigma,l} dV_g = -2\log l(1 + O(\delta^2)) + O(1) + O(\delta^2)(|\log l| + |\log \delta|)$$

Hence letting $\delta$ tends to zero we get the desired conclusion.

*Proof of* (31)
The proof of $(ii)$ comes from direct calculations.
.

*Proof of* (32)
    The proof of this inequality relies on showing the following two pointwise estimates on the gradient of $\varphi_{l,\sigma}$

$$(36) \qquad |\nabla \varphi_{l,\sigma}(y)| \le Cl; \qquad \text{for every } y \in \Sigma,$$

where $C$ is a constant independent of $\sigma$ and $l$, and

$$(37) \qquad |\nabla \varphi_{l,\sigma}(y)| \le \frac{4}{d_{min}(y)} \qquad \text{where} \qquad d_{min}(y) = \min_{i=1,\dots,m} d(y, x_i).$$

    For proving (36) we notice that the following inequality holds

$$(38) \qquad \frac{l^2 d(y, x_i)}{1 + l^2 d^2(y, x_i)} \le Cl, \qquad i = 1, \dots, m,$$

where $C$ is a fixed constant (independent of $l$ and $x_i$). Moreover we have

$$(39) \qquad \nabla \varphi_{l,\sigma}(y) = -2l^2 \frac{\sum_i t_i(1 + l^2 d_i^2(y))^{-3} \nabla_y (d_i^2(y))}{\sum_j t_j(1 + l^2 d_j^2(y))^{-2}}.$$

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Using the fact that $|\nabla_y(d_i^2(y))| \leq 2d_i(y)$ and inserting (38) into (40) we obtain immediately (36). Similarly we find

$$|\nabla \varphi_{l,\sigma}(y)| \leq 4l^2 \frac{\sum_i t_i(1 + l^2 d_i^2(y))^{-3} d_i(y)}{\sum_j t_j(1 + l^2 d_j^2(y))^{-2}} \leq 4l^2 \frac{\sum_i t_i(1 + l^2 d_i^2(y))^{-2} \frac{d_i(y)}{l^2 d_i^2(y)}}{\sum_j t_j(1 + l^2 d_j^2(y))^{-2}}$$

$$\leq 4 \frac{\sum_i t_i(1 + l^2 d_i^2(y))^{-2} \frac{1}{d_{min}(y)}}{\sum_j t_j(1 + l^2 d_j^2(y))^{-2}} \leq \frac{4}{d_{min}(y)},$$

which is (37).
From we infer that

$$(40) \qquad \int_{\cup_{i=1}^k B_{x_i}(\frac{1}{l})} |\nabla_g \varphi_{\sigma,l}|^2 \leq Ck;$$

for some constant depending only on $\Sigma$.
Now for every $i = 1, \cdots, k$ we set

$$(41) \qquad B_i = \{y \in \Sigma \ d(y, x_i) = d_{min}(y)\}$$

and we have

$$(42) \qquad \int_{\Sigma \cup_{i=1}^k B_{x_i}(\frac{1}{l})} |\nabla_g \varphi_{\sigma,l}|^2 dV_g \leq \sum_{i=1}^k \int_{B_i \setminus B_{x_i}(\frac{1}{l})} |\nabla_g \varphi_{\sigma,l}|^2 dV_g \leq 16 \sum_{i=1}^k \int_{B_i \setminus B_{x_i}(\frac{1}{l})} \frac{1}{d(y, x_i)^2} dV_g(y)$$

$$\leq 32\pi(1 + o_l(1)) \log l + O(1).$$

From this and (40) we deduce (32).

Hence the proof of Claim is complete.
Next using the Claim and the definition of $\text{II}_\rho$ we get

$$(43) \qquad \text{II}_\rho(\varphi_{\sigma,l}) \leq (16k\pi - 2\rho_1 + o_l(1)) \log l + O(1).$$

Thus using the fact that $8k\pi < \rho_1$ we get that

$$(44) \qquad \text{II}_\rho(\varphi_{\sigma,l}) \to -\infty \quad \text{uniformly in } \sigma$$

Hence the proof of $(i)$ is completed.
Now let us show $(ii)$. Firts of all we remark for every given $x$, the trivial convergence holds

$$(45) \qquad (\frac{l}{1 + l^2 d(x, y)^2})^2 \rightharpoonup \pi \delta_x$$

in the weak sens of measure. Hence using the definition of $\varphi_{\sigma,l}$ one check easily that

$$(46) \qquad e^{\varphi_{\sigma,l}} \rightharpoonup \sigma.$$

Onb the other hand from $(i)$ we have that the following composition for large $l$

$$(47) \qquad T_l = \Psi \circ \Phi_l$$

is well defined. Moreover from (46) and the continuity of $\Psi$ we infer that for $\bar{l}$ large $T_l$ is an homotopy beetween $\Psi \circ \Phi_{\bar{l}}$ and identity on $\Sigma_k$. Thus the proof of $(ii)$ is complete. Hence the proof of the proposition is concluded.

∎

## 3.4 Topological argument

In this Subsetion we perform the topological argument in order to produce solutions. We will employ a min-max scheme based on the topological cone $C_k$ (for precise definition see below) over $\Sigma_k$. As anticipated in the introduction, we then define a modified functional $II_{t\rho_1,t\rho_2}$ for which we can prove existence of solutions in a dense set of the values of $t$. Following an idea of Struwe, this is done proving the a.e. differentiability of the map $t \mapsto \alpha_{t\rho}$, where $\alpha_{t\rho}$ is the minimax value for the functional $II_{t\rho_1,t\rho_2}$ given by the scheme.

Let $C_k$ be the topological cone over $C_k$, see. First, let $L$ be so large that Proposition 3.8 applies with $\frac{L}{4}$, and choose then $\Phi$ such that Proposition 3.9 applies for $L$. Fixing $L$ and $\Phi$, we define the class of maps

$$(48) \qquad \Pi_{\Phi_{\bar{l}}} = \left\{ \pi : C_k \to H^1(\Sigma) \; : \; \pi \text{ is continuous and } \pi|_{\Sigma_k(=\partial K_k)} = \Phi_{\bar{l}} \right\}.$$

Then we have the following properties.

**Lemma 3.10** *The set $\Pi_\Phi$ is non-empty and moreover, letting*

$$\alpha_\rho = \inf_{\pi \in \Pi_\Phi} \sup_{m \in C_k} II_{\rho_1,\rho_2}(\pi(m)), \qquad \text{there holds} \qquad \alpha_\rho > -\frac{L}{2}.$$

PROOF. To prove that $\Pi_{\Phi_{\bar{l}}} \neq \emptyset$, we just notice that the following map

$$(49) \qquad \overline{\pi}(\sigma, t) = t\Phi_{\bar{l}}(\sigma); \qquad \sigma \in \Sigma_k, t \in [0,1] \quad ((\sigma,t) \in C_k)$$

belongs to $\Pi_{\Phi_{\bar{l}}}$. Assuming by contradiction that $\alpha_\rho \leq -\frac{L}{2}$, there would exist a map $\pi \in \Pi_{\Phi_{\bar{l}}}$ with $\sup_{\tilde{\sigma} \in C_k} II(\pi(\tilde{\sigma})) \leq -\frac{3}{8}L$. Then, since Proposition 3.8 applies with $\frac{L}{4}$, writing $\tilde{\sigma} = (\sigma, t)$, with $\sigma \in \Sigma_k$, the map

$$t \mapsto \Psi \circ \pi(\cdot, t)$$

would be an homotopy in $\Sigma_k$ between $\Psi \circ \Phi_{\bar{l}}$ and a constant map. But this is impossible since $\Sigma_k$ is non-contractible (see Lemma 3.1) and since $\Psi \circ \Phi_{\bar{l}}$ is homotopic to the identity, by Proposition 3.9. Therefore we deduce $\overline{\overline{\Pi}}_{\Phi_{\bar{l}}} > -\frac{L}{2}$. ∎

PROOF OF THEOREM 1.1 We introduce a variant of the above minimax scheme, following [38] and [14]. For $t$ close to 1, we consider the functional

$$II_{t\rho_1,t\rho_2}(u) \quad = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g - t\rho_1 \log \int_\Sigma e^{u-\bar{u}}dV_g - t\rho_2 \log \int_\Sigma e^{-u+\bar{u}}dV_g.$$

Repeating the estimates of the previous sections, one easily checks that the above minimax scheme applies uniformly for $t \in [1 - t_0, 1 + t_0]$ with $t_0$ sufficiently small. More precisely, given $L > 0$ as before, for $t_0$ sufficiently small we have

$$\sup_{\pi \in \Pi_{\Phi_{\bar{l}}}} \sup_{m \in \partial C_k} II_{t\rho_1,t\rho_2}(\pi(m)) < -2L; \quad \alpha_{t\rho} := \inf_{\pi \in \Pi_\Phi} \sup_{m \in C_k} II_{t\rho_1,t\rho_2}(\pi(m)) > -\frac{L}{2};$$

$$(50) \qquad \qquad \text{for every } t \in [1 - t_0, 1 + t_0],$$

where $\Pi_{\Phi_{\bar{l}}}$ is defined in (48).

Next we notice that for $t' \geq t$ there holds

$$\frac{II_{t\rho_1,t\rho_2}(u)}{t} - \frac{II_{t'\rho_1,t'\rho_2}(u)}{t'} = \frac{1}{2}\left(\frac{1}{t} - \frac{1}{t'}\right)\int_\Sigma |\nabla_g u|^2 dV_g \geq 0, \qquad u \in H^1(\Sigma).$$

Therefore it follows easily that also

$$\frac{\alpha_{t\rho}}{t} - \frac{\alpha_{t'\rho}}{t'} \geq 0,$$

namely the function $t \mapsto \frac{\alpha_{t\rho}}{t}$ is non-increasing, and hence is almost everywhere differentiable. Using Struwe's monotonicity argument, see for example [14], one van see that at the points where $\frac{\alpha_{t\rho}}{t}$ is

differentiable $II_{t\rho_1, t\rho_2}$ admits a bounded Palais-Smale sequence at level $\alpha_{t\rho}$, which converges to a critical point of $II_{t\rho_1, t\rho_2}$. Therefore, since the points with differentiability fill densely the interval $[1 - t_0, 1 + t_0]$, there exists $t_n \to 1$ and $u_n \in H^1(\Sigma)$ such that

$$(51) \qquad -\Delta_g u_n = t_n \rho_1 \left( \frac{e^{u_n}}{\int_\Sigma e^{u_n} dV_g} - 1 \right) - t_n \rho_2 \left( \frac{e^{-u_n}}{\int_\Sigma e^{-u_n} dV_g} - 1 \right).$$

At this stage , it is sufficient to apply Proposition 2.5 to get a limit wich is a solution of (4). This conclude the proof. ∎

# References

[1] Bahri A., *Critical points at infinity in some variational problems*, Research Notes in Mathematics, 182, Longman-Pitman, London, 1989.

[2] Bahri, A., Coron, J.M., *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. 41-3 (1988), 253-294.

[3] Bartolucci, D., Tarantello, G., *The Liouville equation with singular data: a concentration-compactness principle via a local representation formula*, J. Differential Equations bf 185 (2002), 161180.

[4] Bartolucci, D., Chen, C. C., Lin, C.-S., Tarantello, G., *Profile of blow-up solutions to mean field equations with singular data*, to appear in Comm. PDE.

[5] Bredon G.E., *Topology and geometry*, Graduate Texts in Mathematics, 139, 1997.

[6] Brezis H., Li Y.Y., Shafrir I., *A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities*, J. Funct. Anal. 115-2 (1993), 344-358.

[7] Brezis H., Merle F., *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions* Commun. Partial Differ. Equations 16-8/9 (1991), 1223-1253.

[8] Caglioti, E., P. L. Lions, C. Marchioro and M. Pulvirenti, A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description, Commun. Math. Phys., 143 (1992), 501-525.

[9] Caglioti, E., P. L. Lions, C. Marchioro and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. Part II*, Commun. Math. Phys., 174 (1995), 229-260.

[10] Chen C.C., Lin C.S., *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*, Comm. Pure Appl. Math. 55-6 (2002), 728-771.

[11] Chen C.C., Lin C.S., *Topological degree for a mean field equation on Riemann surfaces*, Comm. Pure Appl. Math. 56-12 (2003), 1667-1727.

[12] Chen X.X., *Remarks on the existence of branch bubbles on the blowup analysis of equation $-\Delta u = e^{2u}$ in dimension two*, Commun. Anal. Geom. 7-2 (1999), 295-302.

[13] Chen, W., Li C., *Prescribing Gaussian curvatures on surfaces with conical singularities*, J. Geom. Anal. 1-4, (1991), 359-372.

[14] Ding W., Jost J., Li J., Wang G., *Existence results for mean field equations*, Ann. Inst. Henri Poincaré, Anal. Non Linire 16-5 (1999), 653-666.

[15] Djadli Z., *Existence result for the mean field problem on Riemann surfaces of all genus*, preprint.

[16] Djadli Z., Malchiodi A., *A fourth order uniformization theorem on some four manifolds with large total Q-curvature*, C.R.A.S., 340 (2005), 341-346.

[17] Djadli Z., Malchiodi A., *Existence of conformal metrics with constant Q-curvature*, preprint.

[18] Dunne, G., *Self-dual Chern-Simons Theories*, Lecture Notes in Physics, vol. m36, Berlin: Springer-Verlag, 1995.

[19] Hatcher, A., *Algebraic Topology*.

[20] Jeanjean L., Toland, J., *Bounded Palais-Smale mountain-pass sequences*, C. R. Acad. Sci., Paris, S. I, Math. 327, No.1, 23-28 (1998).

[21] Jost J., Lin C.S., Wang, *Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions*, preprint, 2005.

[22] Jost J., Wang G., *Analytic aspects of the Toda system. I. A Moser-Trudinger inequality*, Comm. Pure Appl. Math. 54 (2001) 12891319.

[23] Jost J., Wang G., *Classification of solutions of a Toda system in $\mathbb{R}^2$*, Int. Math. Res. Not., 2002 (2002), 277290.

[24] Joyce G., Montgomery D., *Negative temperature states for the two dimensional guiding-centre plasma*, J. Plasma Phys. 10 (1973), 107-121.

[25] M. K. H. Kiessling, *Statistical mechanics of classical particles with logarithmic interactions*, Comm. Pure Appl. Math., Vol. 46, 1993, pp. 27-56.

[26] Li Y.Y.,*Harnack type inequality: The method of moving planes*, Commun. Math. Phys. 200-2, (1999), 421-444.

[27] Li Y.Y., Shafrir I., *Blow-up analysis for solutions of $-\Delta u = V e^u$ in dimension two*, Indiana Univ. Math. J. 43-4, 1255-1270 (1994).

[28] Li J., Li Y., *Solutions for Toda systems on Riemann surfaces*, preprint, arXiv:math.AP/0504384.

[29] Lucia M., Nolasco M., SU(3) *Chern-Simons vortex theory and Toda systems*, J. Diff. Eq. 184-2 (2002), 443-474.

[30] Malchiodi, A., *Compactness of solutions to some geometric fourth-order equations*, J. Reine Angew. Math., to appear.

[31] Malchiodi, A., Ndiaye, C,B., *Some exitence reuslts for the Toda system on closed surfaces*, preprint.

[32] Nolasco, M., Tarantello, G., *On a sharp Sobolev-type inequality on two-dimensional compact manifolds*, Arch. Ration. Mech. Anal., 145 (1998), 161195.

[33] Nolasco, M., Tarantello, G., *Vortex condensates for the SU(3) Chern-Simons theory*, Comm. Math. Phys. 213 (2000), no. 3, 599639.

[34] Ohtsuka, H., Suzuki, T., *Meanfiled equation for the equilibruim turbulence and related functional inequality*, Advanced in Differential Equations. 11 (2006), no. 3.

[35] Pointin Y.B., Lundgren T.S., *Statistical mechanics of two dimensional vortices in a bounded container*, Phys. Fluids 19 (1976) 1459-1470

[36] Ricciardi, T., *Mountain pass solutions for a mean field equation from two-dimensional turbulence* preprint.

[37] Shafrir I., *A* Sup + Inf *inequality for the equation* $-\Delta u = Ve^u$ C. R. Acad. Sci. Paris S. I Math. 315-2 (1992), 159-164.

[38] Struwe M., *The existence of surfaces of constant mean curvature with free boundaries*, Acta Math. 160 -1/2(1988), 19-64.

[39] Struwe M., *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*,. Third edition, Springer-Verlag, Berlin, 2000.

[40] Struwe M., Tarantello G., *On multivortex solutions in Chern-Simons gauge theory*, Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. (8)-1, (1998), 109-121.

[41] Yang Y., *Solitons in Field Theory and Nonlinear Analysis*, Springer-Verlag, 2001.